

## Inviscid reacting flow near a stagnation point

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A reacting flow free of molecular transport exhibits noteworthy behaviour in the neighbourhood of a blunt, symmetrical stagnation point. A local analytical study using the Lighthill–Freeman gas model reveals the basic structure of such a flow. Chemical activity is found to affect some, but not all, of the local characteristics of the flow. Unaffected are the pressure and velocity fields near the stagnation point, where the pressure varies quadratically and the velocity varies linearly as in an inert flow. In addition, the stagnation point is found to be in chemical equilibrium for all non-zero reaction rates. On the other hand the density, temperature, and concentration fields are affected by the non-equilibrium reactions. The extent of this effect can be predicted on the basis of a reaction parameter that measures the rate of reaction in terms of the velocity gradient at the stagnation point. A rapidly reacting flow (with reaction parameter greater than unity) approaches the stagnation point with vanishing gradients of density and temperature, whereas a slowly reacting flow approaches with infinite gradients. The flow field is represented mathematically by functions that are regular along the body but non-analytic in the normal direction. Numerical computations support the validity of the local closed-form solution, and provide information on the local effects of the chemical history of the flow.

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### 1. Introduction

One usually thinks of fluid flows as presenting a great variety of patterns, traced by streamlines that curve around bodies and coil in their wakes. However, under the eye of a sufficiently powerful glass most regions of most continuum flows dissolve into a uniform stream. Embedded in this local monotone there are, nevertheless, exceptional points that will not give up their identity under any magnification. Stagnation points, points of flow separation, and the eyes of vortices are some of these exceptional points that retain their character and thus allow for highly localized study, free of the intricacies of the global flow. This results in simple local solutions, often expressible in terms of elementary functions. Such a simple local solution is our aim in the present study of the symmetrical flow of an inviscid reacting gas in the neighbourhood of a blunt stagnation point. We shall retain the word *inviscid* to denote a fluid that is free of the

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molecular transport processes giving rise to viscosity, diffusion, and heat conduction.

The basic study of inviscid stagnating flow seems to have stopped with the now classical solution of potential theory, characterized by linear velocity components and quadratic pressure. To be sure, the literature contains a number of papers on stagnation-point boundary layers including the effects of molecular transport, radiation, and so on, but the indispensable companion—the inviscid external flow—has been generally neglected. A number of questions can be raised on this subject, among which a particularly relevant one concerns the combined effects of compressibility and finite reactions in the external flow, because compressible reacting flows are known to have a peculiar behaviour near the stagnation point. If the chemical reactions are fast compared with a characteristic flow time, the approach to stagnation appears to be smooth, but if the reactions are slow this is not the case. The slowly reacting fluid is delivered to the vicinity of the stagnation point in a state far removed from equilibrium, and then—as the flow speed vanishes—the chemical reactions eventually drive the flow towards equilibrium with drastic changes in the thermodynamic state of the gas. One is left wondering whether the fluid will ultimately reach equilibrium under any reaction rate, or even whether a flow that is zooming towards equilibrium will retain the basic dynamics of the classical potential flow. Such questions have not found a definite answer in the various numerical analyses now available, but seem rather to belong in the province of the closed-form local solution, upon which we embark.

## 2. Local stagnation-point solution

We consider either plane symmetric or axisymmetric flow of an inviscid chemically reacting gas impinging upon a smooth body, and seek an asymptotic description of the flow in the immediate vicinity of the stagnation point.

The choice of co-ordinate system is crucial, and merits some discussion. In general, a local solution of an elliptic system of equations should be sought in polar co-ordinates. However, if the solution is analytic, it can be found more easily as a multiple Taylor-series expansion in rectangular co-ordinates. The present analysis was actually carried out first in polar co-ordinates (Conti & Van Dyke 1966), which confirmed earlier evidence that a reacting flow is never analytic at stagnation. The solution is however semi-analytic, in the sense that it has a regular Taylor-series expansion in the direction tangent to the body, but is non-analytic in the normal direction.

We therefore simplify the exposition by using rectangular co-ordinates, oriented as shown in the upper half of figure 1. Rectangular co-ordinates provide the substantial advantage that we must solve only algebraic rather than differential equations. The price we pay for this simplification is that at one stage we have to appeal to the analysis in polar co-ordinates (which the interested reader can reconstruct) for the result that a certain possible higher-order term is absent.

Polar and rectangular co-ordinates share the defect that they carry the non-analyticity into the fluid instead of letting it be swept around the surface of the

body. In higher approximations, this would lead to the compounding of spurious singularities. The remedy is to use co-ordinates that conform to the surface. Thus we would ultimately strain the polar co-ordinates to make  $\theta = \pm \pi$  describe the body, or replace the rectangular co-ordinates by 'boundary-layer' co-ordinates (figure 1) with  $s$  measured along the surface and  $n$  normal to it.

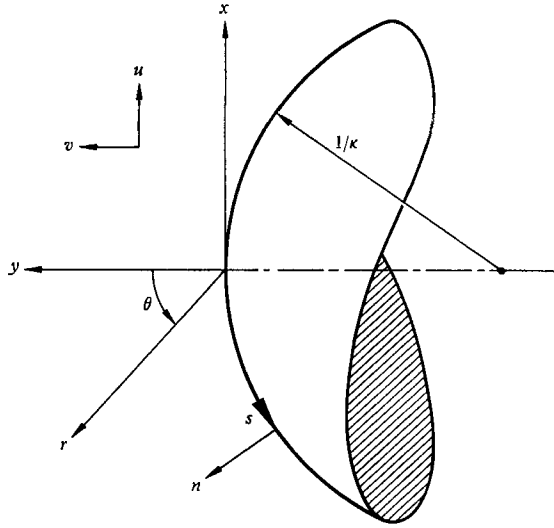


FIGURE 1. Alternative co-ordinate systems for local analysis.

For a simple model of reacting gas we adopt Lighthill's ideal dissociating gas (Lighthill 1957), with Freeman's equation for the rate of reaction (Freeman 1958). The governing conservation equations are

$$\text{mass: } (x^j \rho u)_x + (x^j \rho v)_y = 0, \tag{1}$$

$$\text{x-momentum: } uu_x + vv_y + p_x/\rho = 0, \tag{1a}$$

$$\text{y-momentum: } uv_x + vv_y + p_y/\rho = 0, \tag{1b}$$

$$\text{energy: } h + \frac{1}{2}(u^2 + v^2) = h_0, \tag{1c}$$

$$\text{species: } u\alpha_x + v\alpha_y = (\alpha^* - \alpha)/\tau, \tag{1d}$$

where  $j = 0$  for plane flow and  $j = 1$  for axisymmetric flow.

The velocity components  $u$  and  $v$  are as shown in figure 1;  $p, \rho, T$  and  $h$  stand for pressure, density, temperature and enthalpy, respectively, and  $\alpha$  is the degree of dissociation (atom mass-fraction). The fictitious, local-equilibrium degree of dissociation  $\alpha^*(\rho, T)$  and the reaction time  $\tau$  are defined below. Under our present gas model the thermodynamic variables are further related by

$$\text{thermal state: } p = R(1 + \alpha)\rho T, \tag{1e}$$

$$\text{caloric state: } h = R[(4 + \alpha)T + \theta_a \alpha], \tag{1f}$$

$$\text{mass action: } \frac{\alpha^{*2}}{1 - \alpha^*} = \frac{\rho_a}{\rho} \exp(-\theta_a/T), \tag{1g}$$

$$\text{reaction time: } \tau = \frac{\rho_a}{\Gamma \rho^2 \alpha^*/(1 - \alpha^*) + \alpha}. \tag{1h}$$

Here  $R$  is the gas constant,  $\rho_d$  and  $\theta_d$  are the characteristic density and temperature for the ideal dissociating gas and  $\Gamma$  is Freeman's simplified reaction-rate constant. We have then nine equations for the nine dependent variables  $u, v, p, \rho, T, h, \alpha, \alpha^*$ , and  $\tau$ . It is convenient to reduce their number to eight by introducing the stream function to satisfy the mass-conservation equation:

$$\psi_y = x^j \frac{\rho}{\rho_0} u, \quad \psi_x = -x^j \frac{\rho}{\rho_0} v, \tag{2}$$

where  $\rho_0$  is a reference density.

The local boundary conditions for a symmetrical body in symmetric flow are then

$$\text{symmetry: } \psi = 0 \quad \text{at} \quad x = 0, \tag{3a}$$

$$\text{tangency: } \psi = 0 \quad \text{at} \quad y = -\frac{\kappa}{2}x^2 + \dots, \tag{3b}$$

where  $\kappa$  is the initial curvature of the body (figure 1). These conditions are homogeneous and incomplete, so that as usual in local analysis of an elliptic system we are dealing with an eigenvalue problem. The solution will contain indeterminate constants that can be found only by fitting it asymptotically to a global solution, which in this case must be calculated numerically.

*Leading terms*

We assume that the solution is analytic in the tangential co-ordinate  $x$ , and expand each dependent variable in even or odd powers according to its symmetry:

$$\left. \begin{aligned} \psi &= \frac{1}{1+j} x^{1+j} [\psi_1(y) + x^2 \psi_2(y) + \dots], \\ p &= p_1(y) + x^2 p_2(y) + \dots, \\ T &= T_1(y) + x^2 T_2(y) + \dots, \end{aligned} \right\} \tag{4}$$

with expansions similar to the last two for the symmetrical variables  $\alpha, \tau, \rho, h$ , and  $\alpha^*$ . Substituting into the full equations (1) and equating like powers of  $x$  yields for the leading terms in each equation

$$p_2 = \frac{\rho_0^2}{2} \left[ \frac{\psi_1}{1+j} \left( \frac{\psi_1'}{\rho_1} \right)' - \frac{\psi_1'^2}{(1+j)^2 \rho_1} \right], \tag{5a}$$

$$\rho_0^2 \psi_1 \left( \frac{\psi_1'}{\rho_1} \right)' + p_1' = 0, \tag{5b}$$

$$h_1 = h_0 - \frac{\rho_0^2}{2} \left( \frac{\psi_1'}{\rho_1} \right)^2, \tag{5c}$$

$$-\frac{\rho_0}{\rho_1} \psi_1 \alpha_1' = \frac{\alpha_1^* - \alpha_1}{\tau_1}, \tag{5d}$$

$$p_1 = R(1 + \alpha_1) \rho_1 T_1, \tag{5e}$$

$$h_1 = R[(4 + \alpha_1) T_1 + \theta_d \alpha_1], \tag{5f}$$

$$\frac{\alpha_1^{*2}}{1 - \alpha_1^*} = \frac{\rho_d}{\rho_1} \exp(-\theta_d/T_1), \tag{5g}$$

$$\frac{1}{\tau_1} = \frac{\Gamma}{\rho_d} \rho_1^2 \left( \frac{\alpha_1^*}{1 - \alpha_1^*} + \alpha_1 \right). \tag{5h}$$

The symmetry condition (3a) has been incorporated into the expansions, and the remaining tangency boundary condition (3b) gives

$$\psi_1(0) = 0. \tag{6}$$

The density will have a finite value at the stagnation point, which we henceforth identify with the reference density  $\rho_0$ . Accordingly, we set

$$\rho_1(y) = \rho_0 + O(y^n) \quad \text{as } y \rightarrow 0, \tag{7a}$$

where the exponent  $n$  of the secondary term will be found later. Another quantity whose proper functional form we need is the second-order coefficient for the pressure which, at the stagnation point, is related to the velocity gradient by  $p_2 = \frac{1}{2} \rho_0 (\partial u / \partial x)^2$ . It is well known that the stagnation-point velocity gradient is finite for blunt bodies and we therefore expand as in (7a),

$$p_2(y) = p_{2_0} + O(y^n). \tag{7b}$$

Then the  $x$ -momentum equation (5a) gives, with a relative error of  $O(y^n)$ ,

$$(1 + j) \psi_1 \psi_1'' - \psi_1'^2 = 2(1 + j)^2 \frac{p_{2_0}}{\rho_0}. \tag{8}$$

In view of the tangency condition (6), the solution is

$$\psi_1 = Ay + O(y^{1+n}), \quad A^2 = -2(1 + j)^2 \frac{p_{2_0}}{\rho_0}. \tag{9a}$$

Then the  $y$ -momentum and energy equations (5b) and (5c) give

$$p_1 = p_0 - \frac{1}{2} \rho_0 A^2 y^2 + O(y^{2+n}), \tag{9b}$$

$$h_1 = h_0 - \frac{1}{2} A^2 y^2 + O(y^{2+n}), \tag{9c}$$

where subscript zero denotes a stagnation-point value.

To complete the first cycle of solution giving leading terms in the thermodynamic and dynamic variables, we expand all thermodynamic variables as in (7a) and substitute into the last five equations of (5), which, with an obvious extension of notation, yields for the leading terms

$$\frac{\alpha_0^* - \alpha_0}{\tau_0} = 0, \tag{9d}$$

$$p_0 = R(1 + \alpha_0) \rho_0 T_0, \tag{9e}$$

$$h_0 = R[(4 + \alpha_0) T_0 + \theta_d \alpha_0], \tag{9f}$$

$$\frac{\alpha_0^{*2}}{1 - \alpha_0^*} = \frac{\rho_d}{\rho_0} \exp(-\theta_d / T_0), \tag{9g}$$

$$\frac{1}{\tau_0} = \frac{\Gamma}{\rho_d} \rho_0^2 \alpha_0 \frac{2 - \alpha_0}{1 - \alpha_0}. \tag{9h}$$

The system (9) is indeterminate, with the equations outnumbered by two unknowns,  $p_{2_0}$  and a thermodynamic variable, say  $p_0$ . These quantities cannot be obtained elsewhere within a purely local solution, and recourse must be had to a global solution that includes details of the distant flow and the intervening chemistry. However, if we anticipate that the present solution is valid in some neighbourhood of the stagnation point, equations (9) do answer some of our original questions. Equations (9a) and (9b) show that sufficiently close to the stagnation point the basic dynamics of the classical potential flow is still valid, since the linear velocity and quadratic pressure are not distorted by the chemical reactions. Equation (9d) shows that for other than identically frozen flow ( $\tau_0 = \infty$ ) the fictitious-equilibrium and actual degree of dissociation are equal at the stagnation point, and, since the former obeys the law of mass action (9g), so does the latter. Thus the stagnation point is established to be in chemical equilibrium for any finite reaction rate. We observe in passing that for a sharp-nosed body with a fractional-power velocity, so that  $\psi_1 = By^n$  with  $0 < n < 1$ , (7b) should be replaced by  $p_2 = p_{2_1}y^{n-1} + \dots$  and (5d) would yield

$$(\alpha_0^* - \alpha_0)/\tau_0 B = \text{const.}$$

instead of (9d). It follows that in this case the stagnation point is out of equilibrium by an amount dependent upon the flow and chemical kinetics.

#### *Secondary terms for the thermodynamic variables*

Next we seek secondary terms for the thermodynamic variables by assuming, subject to modification in special cases, the expansions

$$\left. \begin{aligned} \rho_1 &= \rho_0(1 + C_\rho y^n + \dots), \\ \alpha_1 &= \alpha_0(1 + C_\alpha y^n + \dots), \\ \alpha_1^* &= \alpha_0(1 + C_\alpha^* y^n + \dots), \\ T_1 &= T_0(1 + C_T y^n + \dots), \\ \tau_1 &= \tau_0(1 + C_\tau y^n + \dots). \end{aligned} \right\} \quad (10)$$

Again substituting into equations (5) yields, for the secondary terms

$$(1 - nA\tau_0)C_\alpha - C_\alpha^* = 0, \quad (11a)$$

$$\frac{\alpha_0}{1 + \alpha_0} C_\alpha + C_\rho + C_T = \begin{cases} 0 & (n \neq 2), \\ -\rho_0 A^2/2p_0 & (n = 2), \end{cases} \quad (11b)$$

$$\alpha_0(1 + \theta_d/T_0)C_\alpha + (4 + \alpha_0)C_T = \begin{cases} 0 & (n \neq 2), \\ -(1 + \alpha_0)\rho_0 A^2/2p_0 & (n = 2), \end{cases} \quad (11c)$$

$$\frac{2 - \alpha_0}{1 - \alpha_0} C_\alpha^* + C_\rho - \frac{\theta_d}{T_0} C_T = 0, \quad (11d)$$

which make a determinate linear system for  $C_\alpha$ ,  $C_\alpha^*$ ,  $C_\rho$  and  $C_T$ . The relaxation-time coefficient is given as a function of these by

$$C_\tau = -2C_\rho - \frac{C_\alpha^* + (1 - \alpha_0)^2 C_\alpha}{(2 - \alpha_0)(1 - \alpha_0)}. \quad (12)$$

For  $n = 2$  the system (11) is non-homogeneous and yields a definite solution. For  $n \neq 2$  it admits a non-trivial eigensolution if its determinant vanishes, which fixes  $n$  at the value

$$K = \frac{1/A}{\tau_0} \left\{ 1 - \frac{1 - \alpha_0}{2 - \alpha_0} \left[ \frac{\alpha_0}{1 + \alpha_0} - \frac{\alpha_0}{4 + \alpha_0} \left( 1 + \frac{\theta_d}{T_0} \right)^2 \right] \right\}. \tag{13}$$

We must include both possibilities, and thus the expansions (10) become

$$\left. \begin{aligned} \frac{\alpha_1}{\alpha_0} &= 1 + \left\{ \begin{array}{l} \frac{H}{K-2} y^2 + \dots, \\ Cy^K \end{array} \right. \\ \frac{\alpha_1^*}{\alpha_0} &= 1 + \left\{ \begin{array}{l} \frac{H}{K} \left( 1 + \frac{2}{K-2} Q \right) y^2 + \dots, \\ QCy^K \end{array} \right. \\ \frac{\rho_1}{\rho_0} &= 1 + \left\{ \begin{array}{l} - \left[ \frac{\rho_0 A^2}{2p_0} \frac{3}{4 + \alpha_0} + P \frac{H}{K-2} \right] y^2 + \dots, \\ - PCy^K \end{array} \right. \\ \frac{T_1}{T_0} &= 1 + \left\{ \begin{array}{l} - \left[ \frac{\rho_0 A^2}{2p_0} \frac{1 + \alpha_0}{4 + \alpha_0} + \frac{\alpha_0}{4 + \alpha_0} \left( 1 + \frac{\theta_d}{T_0} \right) \frac{H}{K-2} \right] y^2 \\ \quad + \dots \\ - \frac{\alpha}{4 + \alpha_0} \left( 1 + \frac{\theta_d}{T_0} \right) Cy^K \end{array} \right. \end{aligned} \right\} \tag{14}$$

$$\text{Here} \quad \left. \begin{aligned} P &= \frac{\alpha_0}{1 + \alpha_0} - \frac{\alpha_0}{4 + \alpha_0} \left( 1 + \frac{\theta_d}{T_0} \right), \\ Q &= \frac{1 - \alpha_0}{2 - \alpha_0} \left[ \frac{\alpha_0}{1 + \alpha_0} - \frac{\alpha_0}{4 + \alpha_0} \left( 1 + \frac{\theta_d}{T_0} \right)^2 \right], \\ H &= K \frac{\rho_0 A^2}{2p_0} \frac{1}{1 - Q} \frac{1 - \alpha_0}{2 - \alpha_0} \left[ 1 - \frac{1 + \alpha_0}{4 + \alpha_0} \left( 1 + \frac{\theta_d}{T_0} \right) \right], \end{aligned} \right\} \tag{15}$$

and  $C$  is an unknown constant multiplying the eigensolution. These results are to be understood in the sense that the secondary term is the one in  $y^2$  if  $K > 2$ , but the one in  $y^K$  if  $K < 2$ . The other term is the tertiary one in the ranges  $2 < K < 3$  and  $1 < K < 2$ , respectively, but of some higher order otherwise, because the next cycle will add terms in  $y^3$  and  $y^{1+K}$ .

In equations (14) we see that the reaction parameter  $K$ , proportional to the ratio of flow time ( $1/A$ ) to chemical relaxation time ( $\tau_0$ ), plays a dominant role in the local behaviour of the flow. It ranges from zero (frozen flow) to infinity (equilibrium flow) and allows for a clear-cut division of the flow into two régimes: the régime of fast reactions ( $1 < K \leq \infty$ ) where density, temperature and degree of dissociation approach their final stagnation values with vanishing derivatives (proportional to  $y^{K-1}$ ), and the régime of slow reactions ( $0 < K < 1$ ) where the same variables approach stagnation with derivatives tending to infinity. The dividing case  $K = 1$  represents that special flow where the rate of reaction and the rate of approach are in precise balance to give a finite derivative at the stagnation point. The situation is sketched for the density in figure 2.

We noted that the pressure and enthalpy (9) retain the basic quadratic variation of the incompressible flow. This feature seems firmly rooted in the behaviour of the flow. It is unaffected by the chemical reactions, and it ordinarily impresses quadratic terms upon the higher-order expansions for density, temperature, and so on. However, it gives rise to a special situation when  $K = 2$ ,

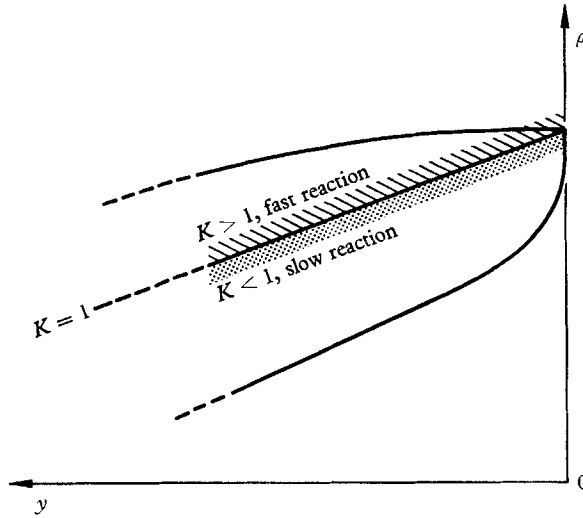


FIGURE 2. Local behaviour of the density near the stagnation point.

since then the pressure-induced quadratic terms combine with their reaction-induced counterparts. The situation bears further attention.

*Confluence of modes at  $K = 2$*

The upper term in each of the equations (14) becomes infinite at  $K = 2$ ; but, because singular behaviour is physically impossible, that divergence must be cancelled by a corresponding singularity in the unknown constant  $C$  that multiplies the lower term. Hence the preceding results are indeterminate at  $K = 2$ . However, we can extract determinate results from them by a simple limiting process, and logarithmic terms emerge. Setting  $K = 2 + \epsilon$  in the expression (14) for  $\alpha_1/\alpha_0$ , for example, and using  $x^\epsilon = 1 + \epsilon \log x + O(\epsilon^2)$ , gives

$$\frac{\alpha_1}{\alpha_0} = 1 + \frac{H}{\epsilon} y^2 + C y^2 (1 + \epsilon \log y + \dots).$$

We can avoid divergence as  $\epsilon \rightarrow 0$  by redefining the unknown constant as

$$C = D - H/\epsilon.$$

Then letting  $\epsilon \rightarrow 0$  yields

$$\frac{\alpha_1}{\alpha_0} = 1 - H y^2 \log y + D y^2 + \dots \quad (K = 2), \tag{16}$$

with corresponding expressions for the other quantities in (14). This shows that terms in  $y^2 \log y$  would have had to be inserted into the assumed expansions



(10) in the case  $K = 2$ . This completes the first half of the second cycle, that is, finding the secondary terms for the thermodynamic variables (and tertiary ones in the range  $1 < K < 3$ , including  $K = 2$  if we count  $y^2 \log y$  and  $y^2$  separately).

*Secondary terms for the dynamic variables when  $K < 1$*

To find secondary terms for the dynamic variables  $p_1$  and  $\psi_1$  we set

$$\left. \begin{aligned} p_1 &= p_0 - \frac{1}{2}A^2\rho_0 y^2(1 + C_p y^n + \dots), \\ \psi_1 &= Ay(1 + C_\psi y^n + \dots). \end{aligned} \right\} \quad (17)$$

Terms with  $n = 1$ , which are proportional to the nose curvature  $\kappa$ , are introduced into even incompressible frozen or equilibrium flow by the next term in  $p_2$ , corresponding to  $n = 1$  in (7b). However, for  $K < 1$  these will be preceded, through coupling with the expressions just calculated, by terms with  $n = K$ ; and we concentrate on this range of slow reactions.

We substitute the expansions (17) into the  $x$ - and  $y$ -momentum equations (5a, 5b), together with  $\rho_1/\rho_0$  from (14). It is at this stage that the present analysis in rectangular co-ordinates yields to the natural superiority of polar co-ordinates. The function  $p_2(y)$  in (4) might contain a term in  $y^K$ , which would enter here. Apparently that term cannot be evaluated within the framework of the present analysis. However, carrying out the solution in polar co-ordinates shows that no such term exists. We choose simply to quote this result without presenting the more complicated analysis in polar co-ordinates. The final results of the substitutions, added to the primary terms already calculated, give for the dynamic variables

$$\psi_1 = Ay \left\{ 1 - \frac{1 - (1+j)K}{(1+K)[2 - (1+j)K]} PCy^K + \dots \right\}, \quad (18)$$

$$p_1 = p_0 - \frac{1}{2}\rho_0 A^2 y^2 \left\{ 1 + \frac{2K[4 + j + (1-j)K - (1+j)K^2]}{(1+K)(2+K)[2 - (1+j)K]} PCy^K + \dots \right\}. \quad (19)$$

Although we have concentrated on the range  $K < 1$ , it is worth while to explain the apparent divergence of these results at  $K = 1$  in axisymmetric flow and at  $K = 2$  in plane flow. The basis of our entire analysis is the eigensolution  $\psi_1 = Ay$ , which may be regarded as describing the irrotational flow of an incompressible inert gas normal to a flat wall. For axisymmetric motion, the second such term is an unknown multiple of  $y^2$  (which describes a *rotational* incompressible flow against a wall). Just as in obtaining (16), we can avoid divergence by redefining the unknown constant multiplying that second eigensolution. This shows that for  $K = 1$  we must add terms in  $y^2 \log y$  to the assumed expansions (17). For plane motion, on the other hand, the second eigensolution is an unknown multiple of  $y^3$ ; so that divergence at  $K = 2$  is avoided by adding terms in  $y^3 \log y$  to (17).

*Critique of a related local solution*

Stulov & Turchak (1966, p. 3) have obtained a local solution similar to the present one as part of their analysis of the flow field about a sphere. They deal with vibrational non-equilibrium rather than molecular dissociation, but nevertheless some of the basic features of both solutions coincide. In particular, the distinctive

fractional-power behaviour of slowly reacting flows is the same. They arrive at a species-conservation equation (3.2) of the same form as the present (5*d*), but at that point proceed to associate the process with a simple relaxation one (of constant relaxation time and forcing function, as evidenced in their solution (3.3)), when in fact the forcing function is coupled to the flow itself. This unnecessarily weakens their argument for equilibrium at the stagnation point, which is a foregone conclusion for a simple relaxation process but not necessarily so for a more general situation. More seriously, in taking that step they suppress the quadratic terms imposed on the forcing function by the pressure and enthalpy, and so miss the upper alternative on the right-hand side of our equations (14). As a consequence, in near-equilibrium flows their temperature and density approach the stagnation value with ever-increasing powers of the distance, when in fact the variation can be no more than quadratic.

### 3. Asymptotic fit to numerical solutions

So far we have pursued the local stagnation-point solution on the grounds of self-consistency. It was clear from the outset that the solution has an eigenvalue character, and so its relevance in any particular case must be found in the context of a larger description of the flow. Specifically, the first-order local solution depends on the global flow for the values of three constants:  $A$ ,  $C$  and, say,  $p_0$ . The thermodynamic state at the stagnation point is unknown to the local solution, but, since we have established that it is an equilibrium state, knowledge of the total enthalpy of the flow reduces the unknown to a single thermodynamic variable, such as  $p_0$ . Our next step will be a numerical analysis of the flow field, searching the stagnation neighbourhood for the behaviour predicted by the local solution.

For convenience, we follow the numerical formulation of Conti (1966) for flow behind a circular-section shock wave of radius  $R_s$ . In present terms, the flow along the stagnation streamline is computed by integrating equations (5), with initial conditions given by the Rankine–Hugoniot (frozen) shock relations, and the assumption that

$$p_2 = -\frac{1}{(y + R_b)^2} p_1 + \frac{1}{2} \frac{1}{y + R_b} \frac{dp_1}{dy}. \quad (20)$$

Here  $R_b$  is the distance between the centre of curvature of the shock wave and the stagnation point, and the form of  $p_2$  is obtained from Conti (1966) through a transformation into the present co-ordinate system. It suffices us to notice that this equation gives  $p_{2_0} = -p_0/R_b^2$ , which bears on the constant  $A$  in the local solution (9*a*).

Equation (20) closes the system (5), which can be rewritten as a sixth-order non-linear system of ordinary differential equations. Our local solution represents a node in the multi-dimensional phase space of that system. This is best illustrated by defining the new variable  $\omega \equiv \alpha_1^* - \alpha_1$ , which vanishes at the stagnation point since there the flow is in equilibrium. Then the local solution gives

$$\alpha_1 = \alpha_0 - \frac{1}{1-Q} \omega + \dots,$$

$$\psi_1 = \frac{A}{[\alpha_0 C(Q-1)]^{1/K}} \omega^{1/K} + \dots,$$

$$p_1 = p_0 - \frac{1}{2} \frac{A^2 \rho_0}{[\alpha_0 C(Q-1)]^{2/K}} \omega^{2/K} + \dots,$$

with corresponding values for the other variables. The situation is outlined in figure 3.

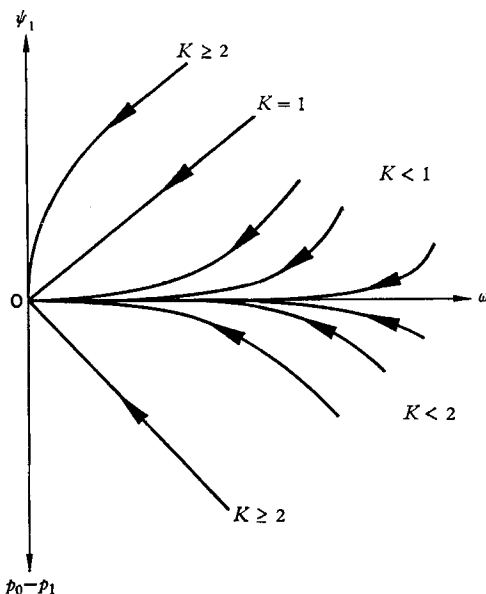


FIGURE 3. Nodal character of solutions near the stagnation point.

One can foresee that those solutions approaching the origin will be drawn to it by the stable singularity, and this proves indeed to be the case in the numerical examples. Calculations carried out with either  $\psi_1$  or  $\omega$  as independent variables always lead the computing machine to the point  $\psi_1 = \omega = 0$ . A family of solutions is generated by assigning particular values to the rate constant  $\Gamma$ , and each member of the family proves to approach the stagnation point with fixed values of  $A$  and  $C$ , and reach a corresponding final state determined by  $p_0$ . As an example of these numerical computations, figure 4 shows the degree of dissociation along the stagnation streamline for axisymmetric flow, and table 1 gives accurate values of  $A$ ,  $C$  and the thermodynamic state at the stagnation point for a range of values of the reaction parameter  $K$ . Flows with planar symmetry behave in essentially the same way as axisymmetric flows, and the case of plane flow with  $K = 1$  is included in the table for illustration. The pursuit of numerical solutions in the immediate vicinity of the stagnation point demands a high computational accuracy, especially at small values of the reaction parameter.

The final state of the stagnant gas varies monotonically with  $\Gamma$  over only a small range, as shown in table 1. This lack of sensitivity to the rate of reaction results, in turn, in a predictable behaviour of the reaction parameter, which is almost proportional to the rate constant  $\Gamma$ . Within its limited range, the degree



of dissociation increases as the reaction becomes slower. This is in accordance with the results of Vinokur (1969), who has made a detailed theoretical study of the state of the gas at the stagnation point. Following his work we interpret the frozen limit as that equilibrium state reached by constant-pressure relaxation from the conditions at the stagnation point in frozen flow, as shown in table 1.

We gain a quantitative idea of how the numerical solutions fall into the form prescribed by the local solution near the body by following the asymptotic behaviour of certain variables. Thus, the quantity  $d\psi_1/dy$  should, according to

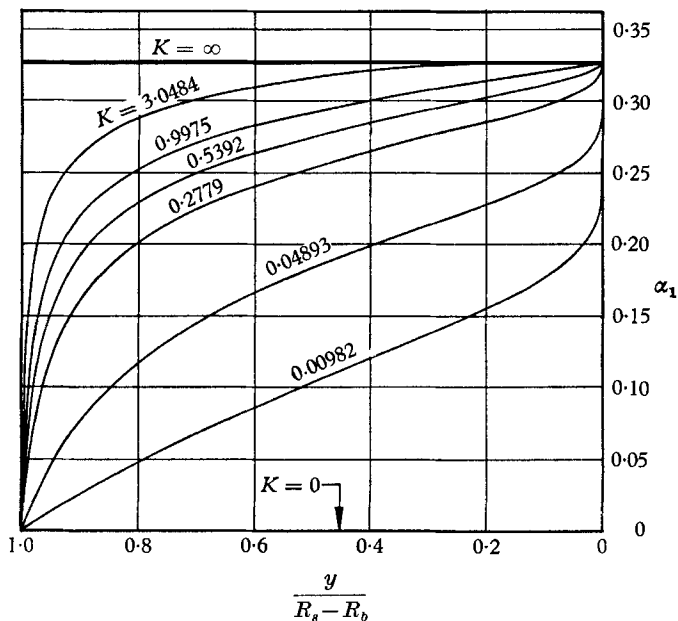


FIGURE 4. Degree of dissociation on the stagnation streamline between the shock wave (abscissa = 1.0) and the body. The flow is axisymmetric, with the parameters of table 1.

the local solution, approach the constant value  $A$  at the stagnation point. Similarly, the quantity  $(d\alpha_1/dy)/(\alpha_0 K y^{K-1})$  should approach the constant value  $C$ . It is advantageous to conduct this search in terms of the departure from equilibrium ( $\omega$ ) rather than the distance to the stagnation point, since then the results are insensitive to the rate of reaction. A typical example is shown in table 2. The clearly asymptotic character of the solution shown there indicates that the local closed-form solution is indeed supported by the numerical examples. The range of validity of the local solution in terms of distance upstream of the stagnation point is very sensitive to the rate of reaction. Thus for  $K \approx 0.5$  the local solution incurs an error in  $\alpha$  of 1% in the first tenth of the shock layer, and a decrease of  $K$  by a factor of ten decreases the distance by a factor of  $10^3$ .

#### 4. Concluding remarks

The local solution developed in §2, and supported by numerical analysis in §3, indicates that the situation in some neighbourhood of a blunt symmetrical stagnation point, to be described in increasing orders of distance from that point, is as follows.

The mathematical structure of the flow is cast by two forces, the quadratic pressure—traceable back to the perfect potential flow—that excites sympathetic integer-power terms in the thermodynamic variables, and the chemical reactions, which introduce fractional-power terms in the axial direction alone. This results in a semi-analytic description of the flow, which is regular along the body but non-analytic in the normal direction.

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$\omega$	$\frac{d\psi_1}{dy} \rightarrow A$	$\frac{d\alpha_1/dy}{\alpha_0 K y^{K-1}} \rightarrow C$
$5 \times 10^{-1}$	10.3	-0.1
$1 \times 10^{-1}$	10.5	-0.04
$1 \times 10^{-2}$	10.68	-0.035
$1 \times 10^{-3}$	10.698	-0.0346
$1 \times 10^{-4}$	10.7003	-0.03460
$1 \times 10^{-5}$	10.70053	-0.034602
$1 \times 10^{-6}$	10.700546	-0.0346018
$1 \times 10^{-7}$	10.7005482	-0.03460177
$1 \times 10^{-8}$	10.70054844	-0.034601772
$1 \times 10^{-9}$	10.700548460	-0.0346017722

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TABLE 2. Asymptotic character of the numerical solution for  $K = 0.04893$  near the stagnation point. All variables are normalized as in table 1.

The stagnation point is found to be in chemical equilibrium for any reaction rate other than zero. This equilibrium state will depend on the chemical history of the stagnating flow, and is therefore out of the province of the local solution. Numerical analysis confirms the independent finding that the stagnation states cluster about the infinite-rate (equilibrium flow) state, which is well removed from the zero-rate (frozen flow) state.

To predict the behaviour of the stagnating flow one needs to know the reaction parameter ( $K$ ), whose precise value can be found only *a posteriori*. However, for the Lighthill-Freeman gas model the reaction parameter is not very sensitive to the chemical history of the flow, because neither is the final state of the gas upon which it depends.

One finds that, confronted with the new situation of non-equilibrium chemical reactions, the stagnating inviscid flow responds by retaining equilibrium at the body, and allowing for drastic departures from it in the normal, but not in the tangential direction. This in turn presents boundary-layer theory with a similar challenge, since it must conform to this external flow but it was not originally devised to accommodate that type of irregularity.

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